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Citation style: Kierat Władysław, Krasińska Stefania. (1972). On the difference equations of two variables. "Prace Naukowe Uniwersytetu Śląskiego w Katowicach. Prace Matematyczne" (Nr 2 (1972), s. 25-33)

PRACE NAUKOWE UNIWERSYTETU ŚLĄSKIEGO W KATOWICACH NR 12

PRACE MATEMATYCZNE II, 1972

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On the difference equations of two variables

INTRODUCTION. In order to find the solutions of partial difference equations with constant coefficients usually the methods of "generating functions" [2] are applied.

In this paper we demonstrate how the discrete operational calculus can be applied to solving partial difference equations with constant coefficients in two discrete variables. In sections 1 and 2 we define the notion of a discrete operator (generalized sequence [4]). In sections 3 and 4 we introduce the translation operator. This operator is the basic operator of the discrete operational calculus. The general solution of difference equation is discussed in sections 5 and 6.

1. THE DISCRETE OPERATORS. Let E be a set of sequences $x = \{x_n\}$ of real numbers such that $x_n = 0$ for $n < N_x$; n ranges over the set of integers. In the set E we introduce the following algebraic operations

$$x^1+x^2 = \{x_n^1\} + \{x_n^2\} = \{x_n^1 + x_n^2\} = x,$$

$$x^1x^2 = \{x_n^1\}\{x_n^2\} = \left\{\sum_{i=-\infty}^{\infty} x_{n-i}^1\, x_i^2\right\} = x.$$

It may easily be verifie dthat the set E with these operations is a commutative ring. Let us denote by P the set of all sequences from the set E such that their first element which is diferent from zero is positive. Then it is possible to show that for every $x \in E$ we have $x \in P$, or $x = 0$, or $-x \in P$, and if $x^1 \in P$ and $x^2 \in P$ then $x^1 + x^2 \in P$ and $x^1x^2 \in P$. Hence, we can introduce in the ring E the relation of order $x^1 > x^2$ if and only if $x^1 - x^2 \in P$.

This relation is a linear order in the ring E ([1] p. 270).

The set E_0 of all elements $x \in E$ such that $x_0 \neq 0$ and $x_n = 0$ for $n \neq 0$ is a subring of the ring E isomorphic to the field of real numbers. The set E is a vector space over the field E_0.

We introduce in E a topology by means of the open intervals $I = \{x : x^1 < x < x^2\}$. With this topology the ring E becomes a linear ordered topological ring.

This topology in a natural way yields a notion of the limit as follows

$$\bigwedge_{\varepsilon \in PN} \bigvee (n > N \rightarrow |x^n - x| < \varepsilon).$$

We have used the absolute value in the usual sense.

The sequence $\omega = \{\omega^n\}$ such that $\omega_1 = 1$ and $\omega_n = 0$ for $n \neq 1$ has the following property

$$\omega x = \omega \{x_n\} = \{x_{n-1}\}.$$

In fact,

$$\omega x = \{\omega_n\} \{x_n\} = \left| \sum_{i=-\infty}^{\infty} x_{n-i} \omega_i \right| = \{x_{n-1}\}.$$

We denote by $\omega^{-1} = \{\omega_n\}$ the sequence such that $\omega_{-1} = 1$ and $\omega_n = 0$ for $n \neq -1$. Hence we obtain

$$\omega^{-1} \omega x = x.$$

We can also verify that $\omega^{-p} = \{x_n\}$, where $x_{-p} = 1$ and $x_n = 0$ for $n \neq -p$. We can easily see, the

$$\omega^p = \{\omega_{n-p}\}.$$

Each element $x = (\ldots 0, x_{N_x,} x_{N_x - 1} \ldots)$ generates the series

$$x = \sum_{i=-\infty}^{\infty} x_i \omega^i = \sum_{i=N_x}^{\infty} x_i \omega^i. \tag{1.1}$$

Let

$$S_n = \sum_{i=-\infty}^{n} x_i \omega^i.$$

We see that the sequence of the partial sums S_n is convergent to x. Hence every element $x \in E$ can be written in the form of series (1.1), which is convergent in the above topology.

We can also verify that

$$\sum_{i=-\infty}^{\infty} x_i^1 \omega^i + \sum_{i=-\infty}^{\infty} x_i^2 \omega^i = \sum_{i=-\infty}^{\infty} (x_i^1 + x_i^2) \omega^i,$$

$$\left(\sum_{i=-\infty}^{\infty} x_i^1 \omega^i \right) \left(\sum_{i=-\infty}^{\infty} x_i^2 \omega^i \right) = \sum_{i=-\infty}^{\infty} \left(\sum_{j=-\infty}^{\infty} x_{i-j}^1 x_j^2 \right) \omega^i.$$

2. DIVISION IN THE RING E. The value of the convolution x^1x^2 at the point n can be written in the following form

(2.1) $$(x^1 x^2)_n = \sum_{i=-\infty}^{\infty} x^1_{n-i} x^2_i = \sum_{i=N_x 2}^{n-Nx^1} x^1_{n-i} x^2_i.$$

We introduce the following notations:

(2.2) $$\{x^1_{n+N_{x^1}}\} = \bar{x}^1 \quad \text{and} \quad \{x^2_{n+N_{x^2}}\} = \bar{x}^2.$$

Then $N_{\bar{x}^1} = N_{\bar{x}^2} = 0$ and

(2.3) $$(x^1 x^2)_n = \sum_{i=0}^{n-Nx^1-Nx^2} \bar{x}^1_{n-i} \bar{x}^2_i.$$

Hence we obtain that E is an integral domain.

The formula (2.3) can be written in the following form:

(2.4) $$\{(x^1 x^2)_n\} = \omega^{N_{x^1}+N_{x^2}} \left| \sum_{i=0}^{\infty} \bar{x}^1_{n-i} \bar{x}^2_i \right|.$$

From (1.1.) we obtain

(2.5) $$x^1 x^2 = \{(x^1 x^2)_n\} = \sum_{i=1}^{\infty} \left(\sum_{i=1}^{n} (\bar{x}^1_{n-i} \bar{x}^2_i) \right) \omega^{n+N_{x^1}+N_{x^2}}$$

Let x^1 and x^2 be fixed elements of E. We seek an element x of E satisfying the equation

(2.6) $$x^2 x = x^1$$

where

$$x^2 \neq 0.$$

The elements x^1 and x^2 can be written in the following form

(2.7) $$x^1 = \omega^{N_{x^1}} \bar{x}^1, \quad x^2 = \omega^{N_{x^2}} \bar{x}^2.$$

It follows from (2.6) and (2.7) that

(2.8) $$\bar{x}^2 x = \omega^{Nx^1 - Nx^2} \bar{x}^1.$$

The element

$$x = \sum_{n=0}^{\infty} \frac{1}{(\bar{x}^2_0)^{n+1}} \begin{vmatrix} \bar{x}^2_0 & 0 & 0 & \ldots\ldots & 0 & \bar{x}^1_0 \\ \bar{x}^2_1 & \bar{x}^2_0 & 0 & \ldots\ldots & 0 & \bar{x}^1_1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \bar{x}^2_n & \bar{x}^2_{n-1} & \bar{x}^2_{n-2} & \cdots & \bar{x}^2_1 & \bar{x}^1_n \end{vmatrix} \omega^{n+Nx^1-Nx^2}$$

satisfies equation (2.6). Hence we obtain that the ring E is a field. The elements of te field E will be called *discrete operators* or *generalized sequences.*

3. THE RING E^+. Let E^+ denote the set of all sequences $x \in E$ such that $N_x \geqslant 0$.

From (2.8) it follows that the set E^+ is a subring of the field E. The set E^+ is a vector space over the field E_0.

We introduce in the space E an endomorphism D by means of the infinite matrix

$$\left|\begin{matrix} 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \dots & & & & \\ \dots & & & & \end{matrix}\right|, \tag{3.1}$$

taking

$$Dx = \left|\begin{matrix} 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \dots & & & & \\ \dots & & & & \end{matrix}\right| \left|\begin{matrix} x_0 \\ x_1 \\ \cdot \\ \cdot \end{matrix}\right|.$$

Observe that

$$\omega D x = x - x_0. \tag{3.2}$$

By induction

$$\omega^k D^k x = x - \omega^{k-1} x_{k-1} - \omega^{k-2} x_{k-2} - \dots - x_0. \tag{3.3}$$

Hence

$$D^k x = \frac{1}{\omega^k} x - \sum_{i=0}^{k-1} \frac{1}{\omega^{k-i}} x_i,$$

where D^k denotes the k-th power of matrix (3.1).

4. THE SPACE E_2^+. By E_2^+ we denote the set of sequences $\{u_{\nu,\mu}\}$ $\nu = 0, 1, \dots;$ $\mu = 0, 1, 2 \dots,$ such that for every fixed ν the sequence $\{u_{\nu,\mu}\}$ belongs to E^+. Using the operation D for the sequence $\{u_{\nu,\mu}\}$ we obtain

$$D u_\nu = D\{u_{\nu,\mu}\} = \frac{1}{\omega}\{u_{\nu,\mu}\} - \frac{1}{\omega} u_{\nu,0}. \tag{4.1}$$

In general, we have

$$D^k u_\nu = D^k\{u_{\nu,\mu}\} = \frac{1}{\omega^k}\{u_{\nu,\mu}\} - \sum_{\varkappa=0}^{k-1} \frac{1}{\omega^{k-\varkappa}} u_{\nu,\varkappa} = \tag{4.2}$$

$$= \frac{1}{\omega^k} u_\nu - \sum_{\varkappa=0}^{k-1} \frac{1}{\omega^{k-\varkappa}} u_{\nu,\varkappa}.$$

We shall write the last formula in the following form

$$\{u_{\nu,\mu+k}\} = D^k \{u_{\nu,\mu}\}. \tag{4.3}$$

In the general case this formula can be written in the following form

$$\{u_{\nu+i,\mu+k}\} = \frac{1}{\omega^k}\{u_{\nu+i,\varkappa}\} - \sum_{\varkappa=0}^{k-1}\frac{1}{\omega^{k-\varkappa}}u_{\nu+i,\varkappa} = \tag{4.4}$$

$$= \frac{1}{\omega^k}u_{\nu+i} - \sum_{\varkappa=0}^{k-1}\frac{1}{\omega^{k-\varkappa}}u_{\nu+i,\varkappa}.$$

5. THE DIFFERENCE EQUATIONS IN TWO VARIABLES. We begin by considering the equation

$$\sum_{i=1}^{m}\sum_{k=0}^{n}\alpha_{ik}u_{\nu+i,\mu+k} = \varphi_{\nu,\mu}. \tag{5.1}$$

Using (4.4) we can write equation (5.1) in the following operational form

$$a_m u_{\nu+m} + \ldots + a_0 u_{\nu=f_\nu}, \tag{5.2}$$

where

$$a_i = a_{in}\frac{1}{\omega^{in}} + \ldots + a_{i0};\quad i = 1, 2, \ldots, m,$$

and

$$f_\nu = \varphi_\nu + \sum_{i=0}^{m}\sum_{k=1}^{n}\sum_{\varkappa=0}^{k-1}\alpha;\quad \frac{1}{\omega^{k-\varkappa}}u_{\nu+i,\varkappa}. \tag{5.3}$$

Ordering expression (5.3) according to the powers of ω we obtain

$$f_\nu = \varphi_\nu + \sum_{\varkappa=0}^{n-1}\frac{1}{\omega^{n-\varkappa}}\sum_{i=0}^{m}\sum_{k=0}^{\varkappa}\alpha_{i,n-\varkappa+k}u_{\nu+i,\varkappa}. \tag{5.4}$$

We can prove this formula similarly as in the case of the Operational Calculus of Mikusiński ([3], p. 291—292).

Now, we shall consider the sequences

$$g_\nu^0 = \sum_{i=}^{m}\alpha_{i,n}\ u_{\nu+i,0},$$

$$g_\nu^1 = \sum_{i=0}^{m}\sum_{k=0}^{1}\alpha_{i,n-1+k}\ u_{\nu+i,k}, \tag{5.5}$$

$$g_\nu^{n-1} = \sum_{i=0}^{m}\sum_{\nu=0}^{n-1}\alpha_{i,1+k}\ u_{\nu+i,k}.$$

If the sequences

$$\{u_{\nu,0}\}, \ldots, \{u_{\nu,n-1}\} \tag{5.6}$$

are given, then we can uniquely determine sequences (5.5). But if we are given sequences (5.5), then not always we can uniquely determine sequences (5.6).

In the case, where sequences (5.6) can not be uniquely determined from equations (5.5), we say that equation (5.1) is *restrictive.*

In the case, where sequences (5.6) can be uniquely determined from equations (5.5), we say that equation (5.1) is *non-restrictive.*

In the non-restrictive case there always exists a solution of equation (5.1) for arbitrary given sequences (5.6) and arbitrary boundary conditions.

In the restrictive case solutions of equation (5.1) satisfying conditions (5.6) not always exist. In this case the conditions on the μ — axis must always ge given in form (5.5).

The following theorem decides which equations of type (5.1) are non-restrictive.

THEOREM. *An equation of type (5.1) is non-restrictive if and only if exactly one of the coefficients a_{in} is different from zero.*

P r o o f. If $a_{i_0,n} \neq 0$, then

$$g_\nu^0 = a_{i_0,n}\, u_{\nu+i_0,0}. \tag{5.7}$$

From the last equality we can uniquely determine the sequence $u_{\nu+i_0,0}$. Each equality of type (5.5) can be written in the following form

$$g_\nu^\varkappa = \sum_{i=0}^{i_0-1}\sum_{k=0}^{\varkappa} a_{i,\,n-\varkappa+k}\, u_{\nu+i,k} + \sum_{k=0}^{\varkappa} a_{i_0,\,n-\varkappa+k}\, u_{\nu+i_0,k} + \sum_{i=i_0+1}^{n}\sum_{k=0}^{\varkappa} a_{i,n-\varkappa+k}\, u_{\nu+i,k}. \tag{5.8}$$

By the assumption of the theorem we obtain

$$g_\nu^\varkappa = a_{i_0,\,n}\, u_{\nu+i_0,\varkappa} + \sum_{i=0}^{m}\sum_{k=0}^{\varkappa-1} a_{i,\,n-\varkappa+k}\, u_{\nu+i,k}. \tag{5.9}$$

Taking $\varkappa = 1$ we have

$$g_\nu^1 = a_{i_0,\,n}\, u_{\nu+i_0,1} + \sum_{i=0}^{m} a_{i,n-1}\, u_{\nu+i,0}. \tag{5.10}$$

From (5.10) we see that if the sequence $\{u_{\nu,0}\}$ can be uniquely determined from (5.5), then also the sequence $\{u_{\nu,1}\}$ can be uniquely determine from (5.5).

From eqquality (5.9) we see immidiately that also the sequences $\{u_{\nu,2}\}, \ldots, \{u_{\nu,n-1}\}$ can be uniquely determined.

Now, we are going to prove that the condition is necessary. In the case, where more than one of the coefficients $a_{i,n}$, $i = 0, \ldots, m$, is not zero, then the problem of determining the sequence $\{u_{\nu,0}\}$ from (5.5) reduces to that of solving the equation

$$\sum_{i=0}^{m} a_{i,n}\, u_{\nu+i,0} = g_\nu^0$$

which has more than one solution. Hence it follws that also the sequences $\{u_{\nu,1}\}, \ldots, \{u_{\nu,n-1}\}$ cannot be uniquely determined from (5.5). Hence equation (5.1) is restrictive.

6. EXAMPLE OF A RESTRICTIVE EQUATION. We shall solve the following difference equation

$$(6.1) \qquad u_{\nu+2,\mu+2} - 2u_{\nu,\mu+1} - u_{\nu,\mu+2} - u_{\nu,\mu} = 2\binom{\nu+\mu+2}{\nu+2} - \binom{\nu+\mu}{\nu+2},$$

with the initial conditions

$$(6.2) \qquad u_{\nu,0} = (-1)^\nu, \quad u_{\nu,1} = (-1)^\nu \frac{2\nu+1}{4} - \frac{1}{4},$$

and with the boundary conditions

$$(6.3) \qquad \{u_{0,\mu}\}, \{u_{\lambda_0,\mu}\}, \text{ where } u_{0,\mu} = \{1\} \text{ and } u_{\lambda_0,\mu} = \left\{\binom{\mu+\lambda_0}{\lambda_0}\right\}$$

for $\mu = 0, 1, 2, \ldots$.

We write equation (6.1) in an operational form

$$(6.4) \qquad \frac{1}{\omega^2} u_{\nu+2} - \frac{2}{\omega} u_\nu - \frac{1}{\omega^2} u_\nu - u_\nu = 2\left\{\binom{\nu+\mu+2}{\nu+2} - \binom{\nu+\mu}{\nu+2}\right\} +$$

$$+ \frac{1}{\omega^2}(u_{\nu+2,0} - u_{\nu,0}) + \frac{1}{\omega}(u_{\nu+2,1} - u_{\nu,1} - u_{\nu,0}).$$

The function $2\left\{\binom{\nu+\mu+2}{\nu+2} - \binom{\nu+\mu}{\nu+2}\right\}$ and the boundary conditions (6.3) also can be writien in the operational form ([4] p. 186)

(6.5) $$2\left\{\binom{\nu+\mu+2}{\nu+2}-\binom{\nu+\mu}{\nu+2}\right\}=\frac{2-\omega^2}{(1-\omega)^{\nu+3}}\,.\quad \{u_{0,\mu}\}=\frac{1}{1-\omega}\quad \text{and}$$

$$\{u_{\lambda_0,\mu}\}=\frac{1}{1-\omega^{\lambda_0+3}}$$ (see [4] p. 181 and 186).

We can easily verify that

(6.6) $$u_{\nu+2,0}-u_{\nu,0}=0 \quad \text{and} \quad u_{\nu+2,1}-u_{\ \ 1}-u_{\nu,0}=0 \text{ for } \nu=0,1,2,\ldots$$

Using (6.5) and (6.6) in (6.4) we obtain

(6.7) $$u_{\nu+2}-(\omega+1)^2u_{\nu}=\frac{\omega^2(2-\omega^2)}{(1-\omega)^{\nu+3}}\,.$$

Let us solve the homogenous equation

(6.8) $$u_{\nu+2}-(\omega+1)^2u_{\nu}=0.$$

The general solution of equation (6.8) has the form

$$\bar{u}_{\nu}=(\omega+1)^{\nu}\,(c_1+(-1)^{\nu}c_2),$$

where c_1 and c_2 belong to E.

We can easily verify that

(6.9) $$u^0_{\ \nu}=\frac{1}{(1-\omega)^{\nu+1}}$$

is a solution of equation (6.7). Hence the general solution of equation (6.6) has the form

(6.10) $$u_{\nu}=\bar{u}_{\nu}+u^0_{\nu}.$$

By the boundary conditions we have

(6.11) $$u_{\nu}=u^0_{\nu}.$$

Solution (6.10) may be written in the usual form

(6.12) $$u_{\nu,\mu}=\binom{\mu+\nu}{\nu}.$$

We can easily verify that function (6.12) does not satisfy conditions (6.2), but satisfies conditions (6.3) and (6.6).

If a function satisfies conditions (6.2), then it also satisfies conditions (6.6). Hence it follows that a solution of equation (6.1) satisfying conditions (6.2) and (6.3) does not exist at all.

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O RÓWNANIACH RÓŻNICOWYCH DWÓCH ZMIENNYCH

Streszczenie

Cząstkowe równania różnicowe o stałych współczynnikach zwykle rozwiązuje się przy pomocy „funkcji tworzących" [2].

W tej pracy pokazujemy jak można stosować dyskretny rachunek operatorów do rozwiązywania równań różnicowych cząstkowych ze stałymi współczynnikami, o dwóch zmiennych. W części 1. i 2. wprowadzamy dyskretny operator (uogólniony ciąg [4]). W części 3. i 4. jest wprowadzony operator przesunięcia.

Ten operator jest podstawowym operatorem dyskretnego rachunku operatorów. Ogólne rozwiązanie różnicowego równania jest przedyskutowane w części 5. i 6.

Oddano do Redakcji 27. 3. 1970.